(#) Ako je $z=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$ izračunati determinantu

$$D=\begin{vmatrix}1 & 1 & 1\\ 1 & z & z^2\\ 1 & z^2 & z\end{vmatrix}.$$

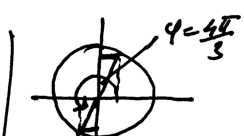
Rj. Prvo izračunajmo vrijednost za $D=\begin{vmatrix}1 & 1 & 1\\ 1 & z & z^2\\ 1 & z^2 & z\end{vmatrix}$.

$$D=\begin{vmatrix}1 & 1 & 1\\ 1 & z & z^2\\ 1 & z^2 & z\end{vmatrix}\ \xrightarrow[III_v-I_v]{II_v-I_v}\ \begin{vmatrix}1 & 1 & 1\\ 0 & z-1 & z^2-1\\ 0 & z^2-1 & z-1\end{vmatrix}=\begin{vmatrix}z-1 & z^2-1\\ z^2-1 & z-1\end{vmatrix}=$$

$$=\begin{vmatrix}z-1 & (z-1)(z+1)\\ (z-1)(z+1) & z-1\end{vmatrix}=(z-1)(z-1)\begin{vmatrix}1 & z+1\\ z+1 & 1\end{vmatrix}=(z-1)(z-1)\left(1-(z+1)^2\right)$$

$$=(z-1)(z-1)\underbrace{\left(1-z^2-2z+1\right)}_{-z^2-2z}=(z-1)(z-1)(-z)(z+2)$$

$$z=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}=$$



$$\cos\frac{4\pi}{3}=-\cos\frac{\pi}{3}$$

$$\sin\frac{4\pi}{3}=-\sin\frac{\pi}{3}$$

$$=-\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}=$$

$$=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

$$z-1=-\frac{1}{2}-\frac{\sqrt{3}}{2}i-1=-\frac{3}{2}-\frac{\sqrt{3}}{2}i$$

$$(z-1)^2=\left(-\frac{3}{2}-\frac{\sqrt{3}}{2}i\right)^2=\frac{9}{4}+\frac{3\sqrt{3}}{2}i+\frac{3}{4}\underbrace{i^2}_{=(-1)}=\frac{6}{4}+\frac{3\sqrt{3}}{2}i=\frac{3}{2}+\frac{3\sqrt{3}}{2}i$$

$$z+2=-\frac{1}{2}-\frac{\sqrt{3}}{2}i+2=\frac{3}{2}-\frac{\sqrt{3}}{2}i$$

$$(-z)(z+2)=\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(\frac{3}{2}-\frac{\sqrt{3}}{2}i\right)=\frac{3}{4}-\frac{\sqrt{3}}{4}i+\frac{3\sqrt{3}}{4}i-\frac{3}{4}i^2=\frac{3}{2}+\frac{\sqrt{3}}{2}i$$

$$D=(z-1)^2\cdot(-z)(z+2)=\left(\frac{3}{2}+\frac{3\sqrt{3}}{2}i\right)\left(\frac{3}{2}+\frac{\sqrt{3}}{2}i\right)=3\sqrt{3}\,i=3\sqrt{3}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$$ traženo rješenje

Ispitati f-ju i nacrtati joj grafik

$$Y = \frac{x-1}{(x^2-2x+4)^2}.$$

Rj: DEFINICIONO PODRUČJE

$(x^2-2x+4)^2 \neq 0$

$x^2-2x+4 \neq 0$

$D = 4-16 < 0$

ova f-ja je uvijek pozitivna

$\mathcal{D}: x \in \mathbb{R}$

$x \in (-\infty, +\infty)$

NULE, PRESJEK SA Y-OSOM, ZNAK

$Y=0$ akko $x-1=0$

$x=1$

$(1,0)$ je nula f-je

$$f(0) = \frac{-1}{4^2} = -\frac{1}{16}$$

$(0, -\frac{1}{16})$ je presjek sa Y-osom

x	$(-\infty, 1)$	$(1, +\infty)$
$x-1$	$-$	$+$
Y	$-$	$+$

znak f-je

PARNOST (NEPARNOST), PERIODIČNOST

$$f(-x) = \frac{-x-1}{(x^2+2x+4)} \neq \pm f(x)$$

PONAŠANJE NA KRAJEVIMA INTERVALA DEFINISANOSTI I ASIMPTOTE

vertikalna asimptota

f-ja nema tački prekida $\Rightarrow$ f-ja nema V.A.

horizontalna asimptota

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x-1}{(x^2-2x+4)^2} = 0 \Rightarrow Y=0 \text{ je H.A.}$$

$x^4 + \ldots$

f-ja nema kosu asimptotu

Poslije ovog koraka počinjemo skicirati grafik f-je.

RAST I OPADANJE

$$Y' = \left(\frac{x-1}{(x^2-2x+4)^2}\right)' =$$

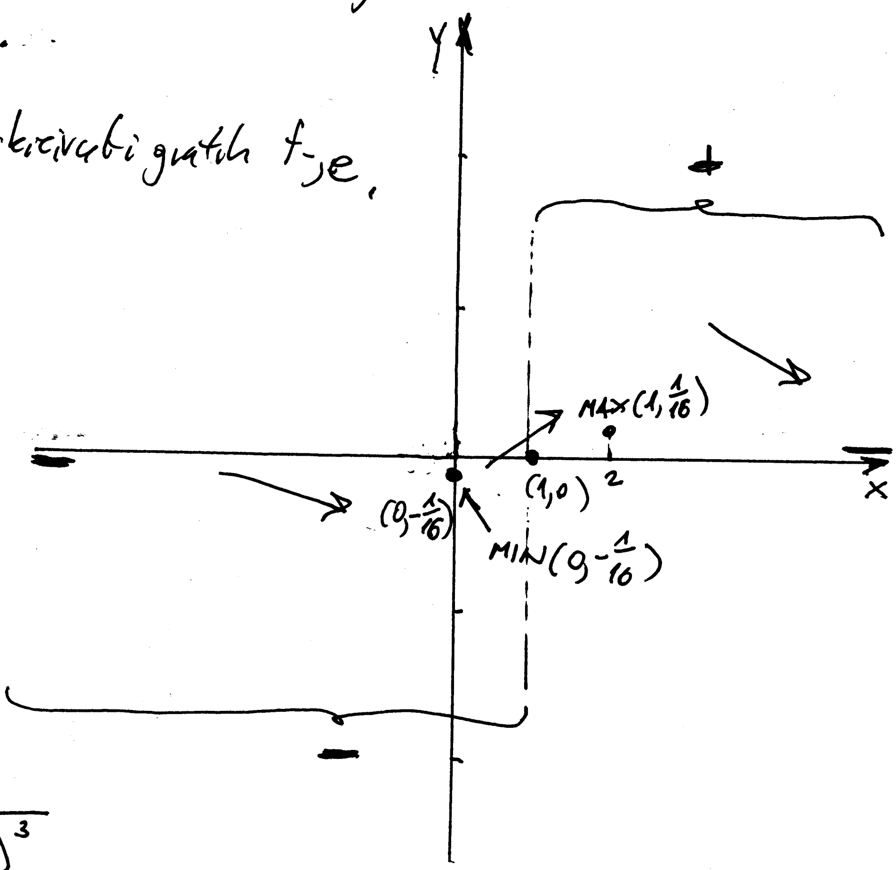
$$= \frac{(x^2-2x+4)^{\cancel{2}} - (x-1)2\cancel{(x^2-2x+4)}\cdot(2x-2)}{(x^2-2x+4)^{\cancel{4}\,3}}$$

$$= \frac{x^2-2x+4-2\cancel{2}(x-1)^2}{(x^2-2x+4)^3}$$

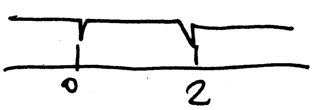
x^2-2x+1

$$= \frac{x^2-2x+\cancel{4}-4x^2+8x-\cancel{4}}{(x^2-2x+4)^3}$$

$$Y' = \frac{-3x^2+6x}{(x^2-2x+4)^3} = (-3)\frac{x(x-2)}{(x^2-2x+4)^3}$$



$y' = 0$ akko $x = 0$ ili $x = 2$



prekid f-je y + nule y'

x	$(-\infty, 0)$	$(0, 2)$	$(2, +\infty)$
y'	−	+	−
y	↘	↗	↘

MIN MAX

$f(0) = -\frac{1}{16}$

$f(2) = \frac{1}{16}$

EKSTREMI F-JE

Na osnovu tabele rasta i opadanja f-ja ima minimum u tački $(0, -\frac{1}{16})$ i maksimum u tački $(2, \frac{1}{16})$.

PREVOJNE TAČKE I INTERVALI KONVEKSNOSTI I KONKAVNOSTI

$$y'' = \left((-3)\frac{x(x-2)}{(x^2-2x+4)^3}\right)' = (-3)\left(\frac{x^2-2x}{(x^2-2x+4)^3}\right)' =$$

$$= (-3)\frac{(2x-2)(x^2-2x+4)^{\cancel{3}} - (x^2-2x)\,3\cancel{(x^2-2x+4)}^{2}(2x-2)}{(x^2-2x+4)^{\cancel{6}\,4}}$$

$$= (-3)\cdot 2\,\frac{(x-1)(x^2-2x+4) - 3(x^2-2x)(x-1)}{(x^2-2x+4)^4} = (-6)\frac{(x-1)\overbrace{(x^2-2x+4-3x^2+6x)}^{-2x^2+4x+4}}{(x^2-2x+4)^4}$$

$$y'' = 12\cdot\frac{(x-1)(x^2-2x-2)}{(x^2-2x+4)^4}$$

$y'' = 0$ akko $x-1=0$ ili $x^2-2x-2=0$

$x = 1$

$D = 4+8 = 12$

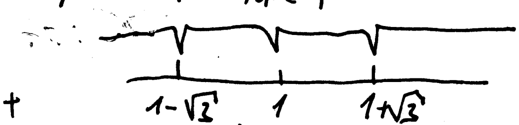
$x_{1,2} = \frac{2 \pm 2\sqrt{3}}{2}$

$x_{1,2} = 1 \pm \sqrt{3}$

$\approx 2{,}73$

$\approx -0{,}73$

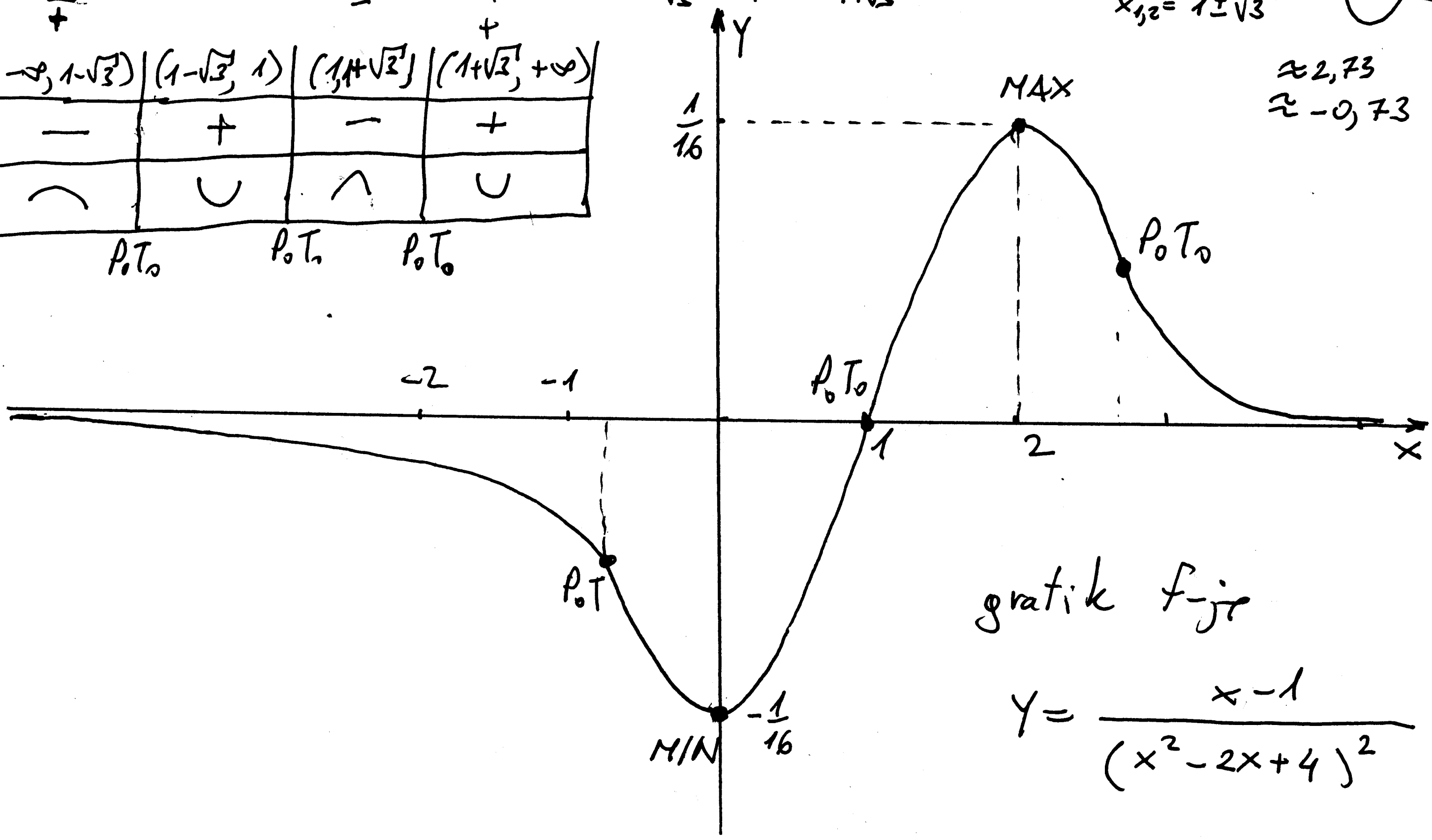
prekidi y + nule y'



$\frac{-}{+}$ $=$ $\frac{+}{-}$ $\frac{+}{+}$

x	$(-\infty, 1-\sqrt{3})$	$(1-\sqrt{3}, 1)$	$(1, 1+\sqrt{3})$	$(1+\sqrt{3}, +\infty)$
y''	−	+	−	+
y	∩	∪	∩	∪

P_oT_o P_oT_o P_oT_o



grafik f-je

$$y = \frac{x-1}{(x^2-2x+4)^2}$$

Izračunati površinu figure koja je određena linijama $y=2x$, $y=\frac{x}{2}$, $y=\frac{2}{x}$.

Rj. Linije $y=2x$ i $y=\frac{x}{2}$ su prave i njih nije teško nacrtati. Problem predstavlja linija $y=\frac{2}{x}$ za koju ne znamo kako izgleda, ispitajmo f-ju $y=\frac{2}{x}$:

DEF. PODR. $\mathcal{D}: x\neq 0$

$x\in\mathbb{R}\setminus\{0\}$

PARN. (NEP.)

$f(-x) = -\frac{2}{x} = -f(x)$ f-ja je neparna

ZNAK

$y>0$ za $x>0$

$y<0$ za $x<0$

NULE, PR. SA Y-osom

$f(0)$ nije definisano

$y\neq 0 \ \forall x\in\mathbb{R}$

VERTIKALNA ASIMPTOTA

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+}\frac{2}{x} = \frac{2}{0+} = +\infty$$
$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-}\frac{2}{x} = \frac{2}{0-} = -\infty$$

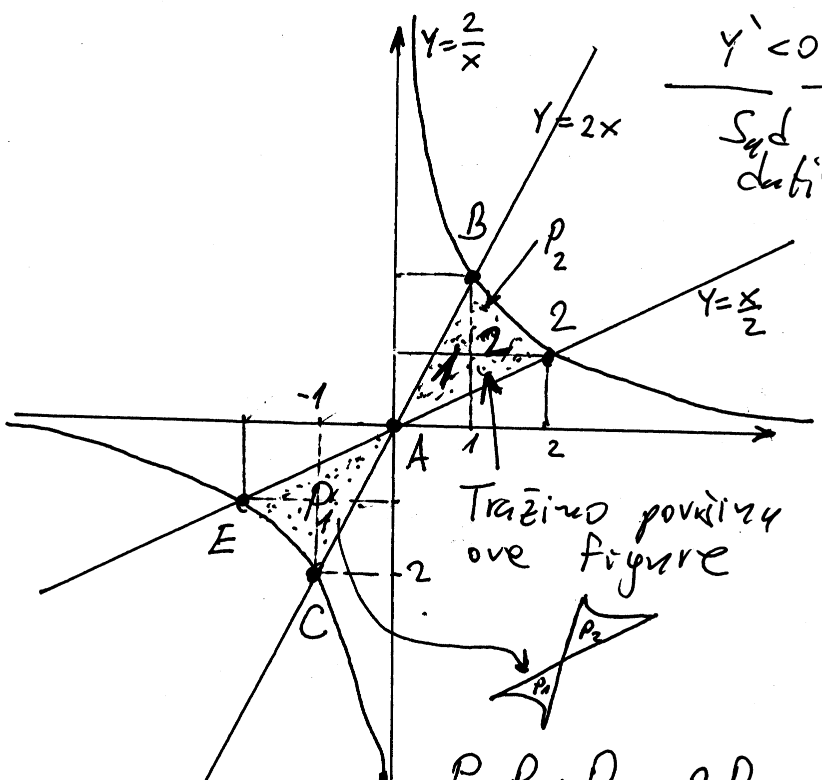
$\Big\}$ $x=0$ je V.A.

HORIZ. ASIMPTOTA

$$\lim_{x\to\pm\infty}\frac{2}{x} = 0 \Rightarrow y=0 \text{ je H.A.}$$

$$y' = \left(\frac{2}{x}\right)' = (2x^{-1})' = -\frac{2}{x^2} \Rightarrow y'\neq 0 \ \forall x\in\mathcal{D}$$

$y'<0 \ \forall x\in\mathcal{D}$ f-ja uvijek opada



Sad pronađimo presječne tačke datih linija

$y=2x$
$y=\frac{1}{2}x$

$2x=\frac{1}{2}x$

$x=0 \Rightarrow y=0$

$A(0,0)$

$y=2x$
$y=\frac{2}{x}$

$2x=\frac{2}{x} \quad /\cdot x \ (x\neq 0)$

$2x^2=2 \quad /:2$

$x^2=1$

$x=\pm 1 \Rightarrow y=\pm 2$

$B(1,1)$

$C(-1,-2)$

$D(2,1)$

$E(-2,-1)$

$y=\frac{x}{2}$
$y=\frac{2}{x}$

$\frac{x}{2}=\frac{2}{x} \quad /\cdot x \ (x\neq 0)$

$\frac{1}{2}x^2=2 \quad /\cdot 2$

$x^2=4 \quad x_{1,2}=\pm 2$

$$P=P_1+P_2 = 2P_2$$

$P_1=P_2$ Površinu P_2 možemo podijeliti na dva dijela

$$I_1 = \int_0^1\left(2x-\frac{x}{2}\right)dx = \int_0^1 \frac{3}{2}x\,dx = \frac{3}{2}\cdot\frac{1}{2}x^2\Big|_0^1 = \frac{3}{4}$$

$$I_2 = \int_1^2\left(\frac{2}{x}-\frac{x}{2}\right)dx = \underbrace{2\ln x\Big|_1^2}_{\ln 2-\ln 1} - \frac{1}{2}\cdot\frac{1}{2}\underbrace{x^2\Big|_1^2}_{4-1} = 2\ln 2 - \frac{3}{4}$$

$\Rightarrow P_2 = I_1+I_2 = 2\ln 2$

$P = 4\ln 2$ tražena površina

(#) Riješiti diferencijalnu jednačinu

$$(2x+y)\,dy = y\,dx + 4\ln y\,dy$$

Rj: Prisjetimo se opšteg oblika difer. jedn. prvog reda:

$y' = f(x)\cdot g(y)$ diferencijalna jednačina sa razdvojenim promjenjivim

$y' = f\left(\frac{y}{x}\right)$ homogena dif. jedn.

$y' = f\left(\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\right)$ dif. jedn. koja se svodi na homogenu

$y' + p(x)y = g(x)$ linearna dif. jedn

$y' + p(x)y = g(x)\,y^n$ Bernul. dif. jedn

$y = x\,f(y') + g(y')$ Lagranžova dif. jedn.

$y = xy' + g(y')$ Klerova dif. jedn.

$$(2x+y)\,dy = y\,dx + 4\ln y\,dy \quad /:dx$$

$$(2x+y)\,y' = y + 4\ln y\; y'$$

$$y = 2xy' + yy' - y'4\ln y$$

$$y = 2xy' + (y-4\ln y)y'$$

odpada Klerova i Lagranžova

$$(2x+y-4\ln y)\,y' = y$$

otpadaju sve preostale

Pokušajmo podijeliti sa dy:

$$(2x+y)\,dy = y\,dx + 4\ln y\,dy \quad /:dy$$

$$2x + y = yx' + 4\ln y \quad /:y$$

$$x' + \frac{4}{y}\ln y - 2x\frac{1}{y} - 1 = 0$$

$$x' - \frac{2}{y}x = 1 - \frac{4}{y}\ln y$$ ovo je linearna dif. jedn po x-u

uvodimo smjenu $x = u\cdot v$

$$x' = u'v + uv'$$

$$u'v + uv' - \frac{2}{y}uv = 1 - \frac{4}{y}\ln y$$

$$u'v + \left(v' - \frac{2}{y}v\right)u = 1 - \frac{4}{y}\ln y$$

a) $v' - \frac{2}{y} v = 0$

$v' = \frac{2}{y} v$

$\frac{dv}{dy} = \frac{2}{y} v$

$\frac{dv}{v} = 2 \frac{dy}{y} \quad / \int$

$\ln v = 2 \ln y$

$\ln v = \ln y^2$

$v = y^2$

b) $u' v = 1 - \frac{4}{y} \ln y$

$u' y^2 = 1 - \frac{4}{y} \ln y$

$\frac{du}{dy} = \frac{1}{y^2} - \frac{4}{y^3} \ln y$

$du = \left(\frac{1}{y^2} - \frac{4}{y^3} \ln y \right) dy$

$u = \int \frac{1}{y^2} dy - \int \frac{4}{y^3} \ln y \, dy$

$$4 \int \frac{1}{y^3} \ln y \, dy = \left| \begin{matrix} u = \ln y & dv = y^{-3} dy \\ du = \frac{1}{y} dy & v = \frac{y^{-2}}{-2} \end{matrix} \right| =$$

$$= 4 \left(-\frac{1}{2} \cdot \frac{1}{y^2} \ln y + \frac{1}{2} \int \frac{dy}{y^3} \right) = -2 \frac{\ln y}{y^2} + 2 \cdot \frac{y^{-2}}{-2}$$

$$= (-2) \frac{1}{y^2} \ln y - \frac{1}{y^2}$$

$$u = -\frac{1}{y} + \frac{2}{y^2} \ln y + \frac{1}{y^2} + c$$

Prema tome

$$x = uv = \left(-\frac{1}{y} + \frac{2}{y^2} \ln y + \frac{1}{y^2} + c \right) y^2$$

$x = -y + 2 \ln y + 1 + c y^2$ opšte rješenje diferencijalne jednačine.